

Notes on General Equilibrium

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Abstract

1 Preliminaries

1.1 Why to study GE

- Applications to Macro, Public economics.
- Learning how mathematical methods should be applied in economics

1.2 Basic math definitions:

Convexity

Closedness

A cube will be denoted by K

Boundness

Compactness

Quasi-concavity.

1.3 Basic Economic Items

m goods (subindex i).

h consumers (subindex j)

A coalition (C) is a subset of the set of consumers, $(1, 2, \dots, h)$.

n firms (subindex f)

Definition of a private ownership economy: $e = (X_j, u_j, w_j, \theta_j, Y_f)_{j=1,2,\dots,h}^{f=1,2,\dots,n}$

1.3.1 Consumers:

The main items:

<i>Name</i>	<i>Notation</i>	<i>Common assumptions</i>
<i>Consumption set</i>	$X_j \subseteq \mathbb{R}_+^m$	<i>Convex, Closed, sometimes Bounded</i>
<i>Preference relation defined on X_j</i>	$\preccurlyeq_j$	<i>Complete, Transitive, Continuous, Non satiated</i>
<i>Utility function</i>	$u_j : X_j \rightarrow \mathbb{R}$	<i>Continuous, Quasi – Concave</i>
<i>Endowments</i>	$w_j \in \mathbb{R}^m$	$w_j \in \text{int}X_j$
<i>Share in profits</i>	$\theta_j \in \mathbb{R}^n$	<i>non negative</i>

Remark 1 *Utility functions and preferences are, under general assumptions, equivalent concepts (Wald, 1943-44, Debreu, 1954).*

Examples:

<i>Name</i>	<i>Example</i>	<i>Example</i>	<i>Example</i>	<i>Example</i>
<i>Consumption set</i>	$\mathbb{R}^m$	$\mathbb{R}^m \cap K$	$\Pi_{i=1}^m x_{ji} \geq a \in \mathbb{R}$	
<i>Preference relation defined on X_j</i>	$\preccurlyeq_j$	<i>Lexicografic</i>		
<i>Utility function</i>	$u_j : X_j \rightarrow \mathbb{R}$	<i>Cobb – Douglas</i>	<i>Linear</i>	<i>CES</i>

Definition 2 *The budget set of consumer j , denoted by $B_j(p, y)$ is the set of consumptions that are feasible and satisfy $px_j \leq pw_j + \sum_{f=1}^n \theta_{jf}py_f$*

1.3.2 Firms

The main items

<i>Name</i>	<i>Notation</i>	<i>Common assumptions</i>
<i>Production set</i>	$Y_f \subseteq \mathbb{R}^m$	<i>Convex, Closed, sometimes Bounded</i>
<i>Transformation Function</i>	$F_f(y_f) = 0$	<i>Continuous, Concave</i>

Remark 3 *If a technology can be represented by a transformation function, it can be represented by a production set but not viceversa.*

Remark 4 *Some components of y_f are positive (outputs), some negative (inputs).*

Remark 5 *Total profits for firm f are py_f*

Remark 6 *When there is only one output (say good k) the transformation function can be written as $y_{fk} = f(y_{f-k})$ where y_{f-k} is the vector of inputs. This is the production function.*

Examples:

<i>Name</i>	<i>Example</i>	<i>Example</i>	<i>Example</i>
<i>Production Function</i>	<i>Cobb – Douglas</i>	<i>Activity Analysis</i>	<i>CES</i>

1.4 The Main Definitions

Definition 7 An allocation (x, y) is a list of consumptions $x_j \in X_j$ $j = 1, 2, \dots, h$ and production vectors $y_f \in Y_f$, $f = 1, 2, \dots, n$

Definition 8 An allocation (x, y) is feasible if for all $i = 1, 2, \dots, m$,

$$\sum_{j=1}^h (x_{ji} - w_{ji}) \leq \sum_{f=1}^n y_{jf}.$$

Definition 9 A feasible allocation (x^P, y^P) is Pareto efficient if there is no feasible allocation (x', y') such that $u_j(x'_j) > u_j(x_j^P)$ for all $j = 1, 2, \dots, h$.

For the next definition we will assume that we are in an exchange economy.

Definition 10 The core is a feasible allocation (x^C) such that there is no coalition C and no x'_j , $j \in C$ such that

- a) $\sum_{j \in C} (x'_{ji} - w_{ji}) \leq 0$.
- b) $u_j(x'_j) > u_j(x_j^C)$ for all $j \in C$

Remark 11 Alternative definitions of Pareto efficiency and the Core are obtained by substituting the inequalities $>$ by $\geq$ with, at least, one inequality strict. If preferences are strictly monotonic, both definitions are equivalent.

Definition 12 A Walrasian (or perfectly competitive) equilibrium is a feasible allocation (x^W, y^W) and a vector of prices p^W such that:

- a) For all $j = 1, 2, \dots, h$ x_j^W maximizes utility on the set $B_j(p^W, y^W)$.
- b) For all $f = 1, 2, \dots, n$, y_f^W maximizes profits $p^W y_f$ for $y_f \in Y_f$.

2 The Two Fundamental Theorems of Welfare Economics

Difference between Strong and Weak Pareto efficiency.

Both equivalent under strict monotonicity.

Make a picture.

From then on, Strong Pareto efficiency will be Pareto efficiency.

Theorem 13 *The Walrasian allocation is weakly Pareto efficient.*

Proof. Only for the case of an exchange economy.

Suppose it is not so, there is a feasible allocation $(x_1, x_2, \dots, x_h)$ such that for all i , $u_j(x_j) > u_j(x_j^W)$. This implies that, at Walrasian prices, this new allocation is not in the budget sets of any consumer, namely

$$p^W x_j > p^W w_j, \forall j = 1, 2, \dots, h.$$

Adding up over consumers we have that

$$\sum_{j=1}^h p^W x_j > \sum_{j=1}^h p^W w_j.$$

Since prices are, by definition non negative the previous inequality contradicts the assumption that the allocation $(x_1, x_2, \dots, x_h)$ is feasible. ■

But sometimes the Walrasian allocation is not Pareto efficient. Picture.

We need the following definition.

Definition 14 *A preference relation $\succsim_i$ is locally nonsatiated if for all $x_i \in X_i$ and ε there is a x'_i at a distance ε such that $x'_i \succ_i x_i$.*

Difference with: Strict Monotonicity and Global Non satiation.

Theorem 15 *If preferences are locally nonsatiated the Walrasian allocation is Pareto efficient.*

Proof. Only for the case of an exchange economy. For the general case see MWG pp. 549-550.

Suppose it is not, so there is a feasible allocation $(x_1, x_2, \dots, x_h)$ such that for all j , $u_j(x_j) \geq u_j(x_j^W)$ with at least one inequality strict. This implies that, at Walrasian prices, this new allocation should be as expensive as the Walrasian one, namely

$$p^W x_j \geq p^W x_j = p^W w_j, \forall j = 1, 2, \dots, h.$$

with strict inequality for those consumers who are actually better off.

Adding up over consumers we have that

$$\sum_{j=1}^h p^W x_j > \sum_{j=1}^h p^W w_j.$$

Since prices are, by definition non negative the previous inequality contradicts the assumption that the allocation $(x_1, x_2, \dots, x_h)$ is feasible. ■

Make a picture.

Theorem 16 *Assume that preferences are strictly monotone. If an allocation $(x_1^P, x_2^P, \dots, x_h^P)$ is Pareto efficient there are prices and a redistribution of endowments for which this allocation is a Walrasian equilibrium provided this equilibrium exists.*

Proof. Only for the case of an exchange economy. For the general case see Maskin and Roberts "On the fundamental theorems of general equilibrium" *Economic Theory* (2008) 35, 233–240.

Give $(x_1^P, x_2^P, \dots, x_n^P)$ as endowments to the consumers. Suppose a Walrasian equilibrium exists. It must be that for all j ,

$$u_j(x_j^W) \geq u_j(x_j^P).$$

Recall that preferences are strictly monotonic and thus both definitions of Pareto efficiency are identical. Thus, if there is a consumer for whom the inequality is strict, we contradict that $(x_1^P, x_2^P, \dots, x_n^P)$ is Pareto efficient. Thus, for all j ,

$$u_j(x_j^W) = u_j(x_j^P).$$

Since consumers can buy their initial endowments this implies that $(x_1^P, x_2^P, \dots, x_n^P)$ is also a Walrasian equilibrium. ■

Make a picture.

Give counterexamples.

A particularly useful way of obtaining Pareto efficient allocations is by considering the following social welfare function

$$W = \sum_{j=1}^h \alpha_j u_j(x_j) \text{ with } (\alpha_1, \alpha_2, \dots, \alpha_h) \in S^h \quad (1)$$

where S^h denotes the $h-1$ dimensional simplex. $(\alpha_1, \alpha_2, \dots, \alpha_h)$ are the weights given to utilities.

Denote by F the set of consumption allocations which are part of a feasible allocation. Define an socially optimal allocation as one maximizing W over F . Then we have the following:

Theorem 17 *If an allocation $(x_1^O, x_2^O, \dots, x_n^O)$ is socially optimal it is Pareto efficient.*

Proof. If not there would exist a feasible allocation such that all agents are better off and this contradicts that $(x_1^O, x_2^O, \dots, x_n^O)$ is socially optimal. ■

Make a picture.

For the converse we need an additional assumption.

The utility possibility set, $U \subseteq \mathbb{R}^h$ is the set of utilities that arise from a feasible allocation. Choose the utility representation such that there are large negative values in U . A set of utilities is optimal if they arise from maximizing (1) over U .

Theorem 18 *If U is convex, any Pareto efficient list of utilities $(u_1^P, u_2^P, \dots, u_h^P)$ is socially optimal for some weights.*

Proof. By Pareto efficiency, $\mathbf{u} \equiv (u_1^P, u_2^P, \dots, u_h^P)$ must belong to the boundary of U . By the supporting hyperplane there is a list $(\lambda_1, \lambda_2, \dots, \lambda_h) \in S^h$ such that $\lambda \mathbf{u} \geq \lambda \mathbf{u}'$ for all $\mathbf{u}' \in U$. Since $\mathbf{u}'$ can be chosen to be negative $\lambda \geq \mathbf{0}$ ■

Make a picture.

Make counterexamples.

Read the FOC approach in MVG.

Problem 19 *Prove Theorems 1 and 2 for the case of an economy with production. Illustrate it with a picture.*

3 Existence of competitive equilibrium 1. Excess demand functions

3.1 Preliminaries

In this approach to the problem of existence of a competitive equilibrium we assume some presumed properties of excess demand functions and see how to solve the problem of solving a system of equations disregarding the microfoundation, although not entirely..

Recall:

m goods (subindex i), h consumers (subindex j), n firms (subindex f)

Total consumption of good i , $\mathbf{x}_i = \sum_{j=1}^h x_{ji}$

Total endowment of good i , $\mathbf{w}_i = \sum_{j=1}^h w_{ji}$

Net production of good i , $\mathbf{y}_i = \sum_{f=1}^n y_{fi}$

Excess demand of good i , $\mathbf{z}_i = \mathbf{x}_i - \mathbf{w}_i - \mathbf{y}_i$

Vector of excess demands $\mathbf{z}$

Vector of prices $\mathbf{p}$

Vector notation: $>>, >, \geq$

3.2 General case

3.2.1 The main assumptions

Let us consider the following 4 assumptions.

F: $\mathbf{z}$ is a single valued mapping of $\mathbf{p}$

Discussion:

1. Restrictive in consumption (piecewise linear indifference curves) and production (no constant returns to scale).
2. Excess demand is hypothetical. It does not reflect actual purchases.

H: $\mathbf{z}(\mathbf{p}) = \mathbf{z}(k\mathbf{p})$ for all $\mathbf{p} > \mathbf{0}$, $k > 0$.

Discussion.

1. The general level of prices is arbitrary. We have to normalize. But we don't know if some good is going to be a free good, so to be on the safe side we consider a simplex:

$$S = \{\mathbf{p} \mid \sum_{j=1}^m p_j = 1, \mathbf{p} > \mathbf{0}\}$$

We exclude negative prices and all prices zero.

2. This property comes from utility and profit maximization.

W: $\mathbf{p} \cdot \mathbf{z} = \mathbf{0}$

Discussion:

It comes from the addition of all budget constraints of consumers with equality (local non satiation)

$$\sum_{i=1}^m p_i (x_{ji} - w_{ji}) = \sum_{f=1}^n \theta_{jf} \mathbf{p} \mathbf{y}_f$$

C: $\mathbf{z}$ is continuous in $\mathbf{p}$

Discussion:

It avoids infinite demands at zero prices and U-shaped cost curves.

3.2.2 Existence of Equilibrium; Preliminaries

Definition 20 $\mathbf{p}^*$ is an equilibrium price vector if $\mathbf{z}(\mathbf{p}^*) \leq 0$

Lemma 21 If W and $\mathbf{z}(\mathbf{p}^*) \leq 0$ $z_i(\mathbf{p}^*) < 0 \rightarrow p_i^* = 0$

Existence proofs are usually based on a fixed point theorem.

Brouwer FPT. Let $F; X \rightarrow X$ be a continuous function with X compact and convex. Then $\exists x^*$ such that $x^* = F(x^*)$

Make the proof for 1 dimension.

Take $M_i(\mathbf{p}) = \max\{0, z_i(\mathbf{p})\}$

This mapping has the following properties for all i

i) $M_i(\mathbf{p}) > 0 \Leftrightarrow z_i(\mathbf{p}) > 0$

ii) $M_i(\mathbf{p}) = 0 \Leftrightarrow z_i(\mathbf{p}) = 0$

iii) $M_i(\mathbf{p}) + p_i \geq 0$.

$M(\mathbf{p})$ represents an adjustment to prices

See AH p. 27 for more examples.

We have to modify it to make sure that it maps S into S .

$$\mathbf{T}(\mathbf{p}) = \frac{\mathbf{p} + \mathbf{M}(\mathbf{p})}{[\mathbf{p} + \mathbf{M}(\mathbf{p})]\mathbf{e}}$$

where $\mathbf{e}$ is a vector of 1's.

3.2.3 Existence of Equilibrium; The Proof

Steps in order to apply Brower fixed point theorem.

1. $T(\cdot)$ fulfils i)-ii)-iii) above.
2. Since this map fulfills the conditions of Brower FPT, $\exists \mathbf{p}^*$ such that

$$\mathbf{p}^* = \frac{\mathbf{p}^* + \mathbf{M}(\mathbf{p}^*)}{[\mathbf{p}^* + \mathbf{M}(\mathbf{p}^*)]\mathbf{e}}$$

3. The fixed point is an equilibrium.

$$\mathbf{p}^*[\mathbf{p}^* + \mathbf{M}(\mathbf{p}^*)]\mathbf{e} = \mathbf{p}^* + \mathbf{M}(\mathbf{p}^*)$$

Solving for $\mathbf{M}(\mathbf{p}^*)$

$$\mathbf{M}(\mathbf{p}^*) = \lambda \mathbf{p}^*$$

where $\lambda = [\mathbf{p}^* + \mathbf{M}(\mathbf{p}^*)]\mathbf{e} - 1$

$$\mathbf{M}(\mathbf{p}^*)\mathbf{z}(\mathbf{p}^*) = \lambda \mathbf{p}^*\mathbf{z}(\mathbf{p}^*) = 0$$

So if $z_i(\mathbf{p}^*) > 0 \rightarrow M_i(\mathbf{p}^*) = 0$ contradicting i). Thus we have proved:

Theorem 22 *Under F, H, W and C there is a competitive equilibrium.*

The proof can be extended to a weaker continuity assumption, see AH Chp. 2, section 8 for details.

3.3 Special case

Is it possible to avoid the fixed point, at least in some special cases?

Greenberg (1977) assumed GS and obtained a very simple proof.

Corchon-Barbolla (1991) made more general assumptions.

"We shall show that if the gross substitutes assumption is replaced by the assumption that good n is a weak gross substitute with the rest of goods and the assumption that the determinant of a Jacobian of excess demand functions is everywhere nonvanishing, we still have a simple proof of the existence of an equilibrium"

Assume that $\mathbf{z}(\mathbf{p})$ is differentiable.

Get rid of good m . By Walras law this must be in equilibrium too.

Assume a boundary condition

Let $\mathbf{L}(\mathbf{p})$ be the $m - 1$ Jacobian of excess demand functions. Assume that

$$\det \mathbf{L}(\mathbf{p}) \neq 0.$$

Consider the following program

$$\text{To min } \sum_{i=1}^{m-1} z_i(\mathbf{p})^2. \quad (2)$$

By bounding suitably the function, this program has a solution which is interior by a boundary assumption. Consider the FOC of (2).

$$\mathbf{L}(\mathbf{p}^*)\mathbf{z}(\mathbf{p}^*) = \mathbf{0} \rightarrow \mathbf{z}(\mathbf{p}^*)$$

So prices solving (2) are equilibrium prices. See the paper for details.

4 Existence of competitive equilibrium 2. Excess demand correspondences.

4.1 Mathematical Preliminaries

In Lecture 3 we neglected

Multi-valued mappings (correspondences).

Microfoundations

We will deal with them in this Lecture. We begin by stating two properties.

Definition 23 A correspondence $\varphi : S \rightarrow T$ is convex-valued if $\forall s \in S$, $\varphi(s)$ is a convex set.

Make a picture.

A function is always convex-valued.

Definition 24 The graph of a correspondence $\varphi : S \rightarrow T$ is closed if

$$[s^q \rightarrow s^0, t^q \rightarrow t^0, t^q \in \varphi(s^q)] \rightarrow [t^0 \in \varphi(s^0)]$$

Make a picture.

Remark 25 If S is compact and $\varphi(\cdot)$ is compact-valued

$$[\text{graph of } \varphi(\cdot) \text{ closed}] \Leftrightarrow [\varphi(\cdot) \text{ is upper-hemicontinuous}]$$

Definition 26 A correspondence $\varphi(\cdot)$ is lower-hemicontinuous if

$$[s^q \rightarrow s^0, t^0 \in \varphi(s^0)] \rightarrow [\exists(t^q), t^q \rightarrow t^0, t^q \in \varphi(s^q)]$$

Remark 27 If $S = [0, 1]$ and $\varphi(s) = (0, 1)$, $\varphi(\cdot)$ is continuous (since it is constant) but the graph is not closed. If $S = \mathbb{R}_+$ and $\varphi(s) = 1/s$ if $s \neq 0$ and 0 otherwise, the graph is closed but is not continuous.

Now we are ready to state Kakutani's fixed point theorem.

Theorem 28 If $\varphi : S \rightarrow S$ is upper-hemicontinuous and convex-valued correspondence and S is compact and convex there is a s^* such that $s^* \in \varphi(s^*)$

Easy proof from Brouwer's FPT.

4.2 Structure of the problem

Recall: m goods (subindex i), h consumers (subindex j), n firms (subindex f).

Vector of excess demands $\mathbf{z}$, Vector of prices $\mathbf{p}$, Vector notation $>>, >, \geq$

We have to deal with simultaneous optimization.

The structure of the problem is similar to a game in normal form with two differences.

1. The budget constraint
2. Feasibility of the allocation

We will remedy 1 by introducing a new item, the budget correspondence.

We will remedy 2 by introducing a fictitious agent, the auctioneer.

Consider an abstract agent with a set of feasible actions Y , a list of environmental variables $x \in X$ and a payoff function $f(y, x)$ and such that $y \in \varphi(x)$.
Let

$$\mu(x) = \{y \in \varphi(x) \mid f(y, x) = \max_{z \in \varphi(x)} f(z, x)\}$$

The following theorem (Berge's maximum theorem) is extremely useful.

Theorem 29 *Let $X, Y \subseteq \mathbb{R}^t$ and $f(\cdot)$ and $\varphi(\cdot)$ be continuous. Then μ is upper hemicontinuous.*

Proof. See Debreu, pp.701-702. ■

This Theorem will be used repeatedly in the exploration of the properties of supply and demand correspondences.

4.3 Producers

We begin by the producers since it is the simpler case.

Assumption 1. $\forall f = 1, \dots, n$, Y_f is compact, convex with $0 \in Y_f$.

Give examples.

Let $\mathbf{y}_f(\mathbf{p})$ be the supply correspondence of firm f , namely

$$\mathbf{y}_f(\mathbf{p}) = \{\mathbf{y}_f \in Y_f \mid \mathbf{p}\mathbf{y}_f \geq \mathbf{p}\mathbf{y}'_f \forall \mathbf{y}'_f \in Y_f\}$$

$0 \in Y_f \rightarrow \mathbf{p}\mathbf{y}_f(\mathbf{p}) \geq 0$ profit shares never cause bankruptcies (not today!!!).

Lemma 30 *Under A.1 $\mathbf{y}_f(\mathbf{p})$ exists.*

Proof. Profits are continuous in $\mathbf{y}_f$. Y_f is compact so Weierstrass theorem implies the result. ■

Lemma 31 *Under A. 1, $\mathbf{y}_f(\mathbf{p})$ is convex valued*

Proof. If $\mathbf{y}_f^1, \mathbf{y}_f^2 \in Y_f$, any convex combination also belongs to Y_f and it yields as much profits as $\mathbf{p}\mathbf{y}_f^1$ ($= \mathbf{p}\mathbf{y}_f^2$). ■

Lemma 32 *$\mathbf{y}_f(\mathbf{p})$ is upper hemicontinuous.*

Proof. By Berge's maximum theorem. ■

So we are done here

4.4 Consumers

The budget correspondence of j is $B_j(\mathbf{p})$ defined as

$$B_j(\mathbf{p}) = \{\mathbf{x}_j \in X_j \mid \mathbf{p}\mathbf{x}_j \leq \mathbf{p}\mathbf{w}_j + \sum_{f=1}^n \theta_{jf} \mathbf{p}\mathbf{y}_f\}$$

The demand correspondence of j is $\mathbf{x}_j(\mathbf{p})$ is defined as

$$\mathbf{x}_j(\mathbf{p}) = \{\mathbf{x}_j \in B_j(\mathbf{p}) \mid u_j(\mathbf{x}_j) \geq u_j(\mathbf{x}'_j) \forall \mathbf{x}'_j \in B_j(\mathbf{p})\}$$

The budget correspondence is not always continuous. Arrow's example.

Assumption 2. $\forall j = 1, \dots, h \exists \hat{x}_j \ll w_j$.

Assumption 3. $\forall j = 1, \dots, h$, X_j is compact and convex.

Lemma 33 *Under A.2-3 the budget correspondence is continuous*

Proof. See Debreu, Lemma 3, p. 707. ■

Let us introduce $\hat{X}_j = \{\mathbf{x}_j \in X_j \mid \mathbf{x}_j \leq \mathbf{w} + \sum_{f=1}^n \mathbf{y}_f\}$.

Assumption 4. $\forall j = 1, \dots, h$ the utility function is continuous, quasi-concave and non satiated in $\hat{X}_j$.

Notice that by Weierstrass theorem the consumer is satiated at X_j !!!

Lemma 34 *Under A.2-3 the demand correspondence $\mathbf{x}_j(\mathbf{p})$ exists*

Proof. Under A. 3 the budget set is never empty. By A. 2 is compact. Given A. 4, Weierstrass theorem proves the result. ■

Lemma 35 $\mathbf{x}_j(\mathbf{p})$ is convex-valued.

Proof. If $\mathbf{x}_j^1, \mathbf{x}_j^2 \in B_j(\mathbf{p})$, any convex combination also belongs to $B_j(\mathbf{p})$ and it yields as much utility as $\mathbf{x}_j^1, \mathbf{x}_j^2$. ■

Lemma 36 *Under A. 2-3-4, $\mathbf{x}_j(\mathbf{p})$ is upper hemicontinuous.*

Proof. By Berge's maximum theorem. ■

4.5 Market equilibrium

Adding up supply and demand correspondences we have shown that

Proposition 37 $\mathbf{z}(\mathbf{p})$ is upper hemicontinuous and convex valued.

Proof. Addition preserves upper hemicontinuity and convex valuedness. ■

We now introduce a new fictitious agent, the auctioneer.

She has actions in S (she chooses a price) and maximizes $\mathbf{p}\mathbf{z}$ for given $\mathbf{z}$.

This maximization mimics the behavior of competitive markets.

Let $\mathbf{p}(\mathbf{z})$ be the auctioneer best reply.

Lemma 38 If $z_i < 0$, $p_i(\mathbf{z}) = 0$.

Let Z be the set of all possible values of $\mathbf{z}$. It is compact and convex

Lemma 39 $\mathbf{p}(\mathbf{z})$ exists, is upper hemicontinuous and convex valued.

Proof. Identical to the properties of $\mathbf{y}_f(\mathbf{p})$. ■

Consider the composition of excess demands and auctioneer's best reply, φ .

$$\varphi : Z \times S \rightrightarrows Z \times S$$

Lemma 40 $\exists(\mathbf{z}^*, \mathbf{p}^*)$ such that $(\mathbf{z}^*, \mathbf{p}^*) \in \varphi(\mathbf{z}^*, \mathbf{p}^*)$.

Proof. By Kakutani's fixed point theorem. ■

And now the final step

Theorem 41 Under A. 1-2-3-4 there is a competitive equilibrium

Proof. To show $\mathbf{z}^* \leq 0$, let r be the $\max_{i=1, \dots, m} \mathbf{z}^*$. If $z_r^* > 0$ all goods with $z_i^* > 0$ obtain $p_i^* = 0$ and this contradicts Walras law. ■

4.6 Addendum

4.6.1 Generalization

Boundness of production and consumption sets can be dispensed with by assuming other properties which imply that the set of feasible allocations is bounded.

4.6.2 Counterexamples.

Make a counterexample of the existence of competitive equilibrium when:

1. Y_f is not bounded
2. Y_f is not closed
3. Y_f is not convex
4. $0 \notin Y_f$
5. $w_j \notin \text{int } X_j$
6. X_j is not closed
7. X_j is not convex
8. The utility function is not continuous,
9. The utility function is not quasi-concave

The only assumptions that are really redundant is that X_j is not bounded and that preferences are not satiated in $\hat{X}_j$.

4.6.3 Generalization:

A Generalized game in normal form...

5 Existence of competitive equilibrium 3. Equilibrium and welfare.

5.1 Mathematical Preliminaries

Recall: m goods (subindex i), h consumers (subindex j), n firms (subindex f).

Vector of excess demands $\mathbf{z}$, Vector of prices $\mathbf{p}$, Vector notation $>>, >, \geq$

Facts. i) The sum of concave functions is concave

ii) The sum of continuous functions is continuous

iii) The sum of quasi-concave functions is not necessarily quasi-concave.

Example of iii): two single peaked functions without and with non negativity constraints.

Consider the following maximization problem

$$\max f(\mathbf{z}) \text{ subject to } g_j(\mathbf{z}) \geq 0, j = 1, 2, \dots, s, \mathbf{z} \in Z. \quad (3)$$

Assuming that $f(\mathbf{z})$ and $g_j(\mathbf{z}) \geq 0, j = 1, 2, \dots, s$ are continuous and Z compact a solution, denoted by $\mathbf{z}^*$, exists (by Weierstrass).

The Lagrangian of (3) is denoted by $\Phi(\mathbf{z}, \mathbf{p})$ where $\mathbf{p}$ is a vector of Lagrange multipliers.

Theorem 42 (Kuhn-Tucker). *Let Z be convex and $f(\mathbf{z}), g_j(\mathbf{z}), j = 1, 2, \dots, s$ be concave. Suppose there is a $\bar{\mathbf{z}}$ such that $g_j(\bar{\mathbf{z}}) > 0, j = 1, 2, \dots, s$ (Slater's condition). Then there is a $\mathbf{p}^* \geq 0$ such that $\Phi(\mathbf{z}^*, \mathbf{p}^*) \geq \Phi(\mathbf{z}, \mathbf{p}^*)$ for all $\mathbf{z} \in Z$.*

5.2 Equilibrium and Welfare

This approach is due to Negishi (1960). Generalized by Arrow and Hahn (1971).

Avoids entirely demand and supply correspondences.

Useful for any solution concept which is PE.

Maintain assumptions 1-4 with quasi-concavity strengthened to concavity (not in AH). This new assumption is referred to as A.4'.

Consider the following maximization for given $\alpha \in S$.

$$\max \sum_{j=1}^h \alpha_j u_j(\mathbf{x}_j) \text{ over } \mathbf{x} - \mathbf{w} \leq \mathbf{y}, \mathbf{y} \in \Pi_{f=1}^n Y_f. \quad (4)$$

Let $\tilde{\mathbf{x}}(\alpha), \tilde{\mathbf{y}}(\alpha)$ be a solution of (4). By arguments similar to Lemma 8, it exists.

Lemma 43 *Under A. 1-4' (4), Kuhn-Tucker theorem applies..*

Proof. The sum of concave functions is concave and the set over which maximization takes place is convex. A. 2 is Slater's condition. ■

Thus :

$$\begin{aligned} \sum_{j=1}^h \alpha_j u_j(\tilde{\mathbf{x}}_j(\alpha)) + \mathbf{p}(\tilde{\mathbf{y}}(\alpha) + \mathbf{w} - \tilde{\mathbf{x}}(\alpha)) &\geq \sum_{j=1}^h \alpha_j u_j(\mathbf{x}_j) + \mathbf{p}(\mathbf{y} + \mathbf{w} - \mathbf{x}) \\ \text{for all } \mathbf{x}, \mathbf{y} \text{ with } \mathbf{x} - \mathbf{w} &\leq \mathbf{y}, \mathbf{y} \in \Pi_{f=1}^n Y_f. \end{aligned} \quad (5)$$

From this it is immediate that

$$\mathbf{p}\tilde{\mathbf{y}}_f(\alpha) \geq \mathbf{p}\mathbf{y}_f \text{ for all } \mathbf{y}_f \in Y_f \quad (7)$$

so firms maximize profits taking the Lagrange multiplier as given. And

$$\text{If } \alpha_j > 0, u_j(\tilde{\mathbf{x}}_j(\alpha)) \geq u_j(\mathbf{x}_j) \text{ for all } \mathbf{x}_j \text{ such that } \mathbf{p}\tilde{\mathbf{x}}_j(\alpha) \geq \mathbf{p}\mathbf{x}_j. \quad (8)$$

Thus consumers are utility maximizers.

By non satiation $\mathbf{p} \neq \mathbf{0}$, so normalize and let $\tilde{\mathbf{p}}(\alpha) = \mathbf{p}(\alpha)/\mathbf{p}(\alpha)\mathbf{e}$.

5.3 Welfare and Equilibrium

Lemma 2 is a proof of the second welfare theorem.

Recall that in our simple proof in Lecture 2 we assumed that a competitive equilibrium exists.

In a representative consumer model we are done.

With many consumers $\tilde{\mathbf{x}}_j(\boldsymbol{\alpha})$ does not fulfil necessarily the budget constraint.

Let $M_i(\mathbf{p}) \equiv \mathbf{p}\mathbf{w}_i + \sum_{f=1}^n \theta_{jf} \mathbf{p}\mathbf{y}_f(\mathbf{p})$ be the income of i .

This function is continuous in $\mathbf{p}$ (difference with the budget set).

We have to produce a "comeback mapping" from allocations to $\boldsymbol{\alpha}'$ s.

Consider the following maximization problem

$$\max \sum_{j=1}^h \alpha_j (M_j(\mathbf{p}) - \mathbf{p}\mathbf{x}_j) \quad \boldsymbol{\alpha} \in S. \quad (9)$$

Interpretation of (9): A fictitious agent chooses $\boldsymbol{\alpha}$ in S in order to maximize the difference between income in a market economy and the value of the consumption allocated by (4).

By Walras law $\sum_{j=1}^h M_j(\mathbf{p}, \mathbf{x}_j) - \mathbf{p}\mathbf{x}_j = 0$.

If an agent spends more than her income she is penalized with a zero alpha.

Let A be the set of feasible allocations as defined in Definition 8.

$$A = \{\mathbf{x} \in \Pi_{j=1}^h X_j, \mathbf{y} \in \Pi_{f=1}^n Y_f \mid \mathbf{x} - \mathbf{w} \leq \mathbf{y}\}.$$

Let $F : A \times S \rightarrow S$ be the mapping induced by (9).

Make a picture.

5.4 Existence of Equilibrium

It's time to put together all the pieces.

By arguments that by now should be familiar we have that

Lemma 44 *Under A.1-2-3-4', $F(\mathbf{p}, \mathbf{x}, \mathbf{y})$, $\tilde{\mathbf{y}}(\alpha)$, $\tilde{\mathbf{x}}(\alpha)$, $\tilde{\mathbf{p}}(\alpha)$ exists, are upper-hemicontinuous and convex valued.*

Proof. Only the upper-hemicontinuity of $\tilde{\mathbf{p}}(\alpha)$ does not follow from Berge's theorem. But the graph is a closed set so we are done. Convex-valuedness follows from the usual arguments. ■

Consider now the collection of mappings $F(\mathbf{p}, \mathbf{x}, \mathbf{y})$, $\tilde{\mathbf{y}}(\alpha)$, $\tilde{\mathbf{x}}(\alpha)$, $\tilde{\mathbf{p}}(\alpha)$.

They map $A \times S$ into $A \times S$

They are upper-hemicontinuous and convex valued.

Lemma 45 *Under A.1-2-3-4', there is a fixed point $\mathbf{p}^*, \mathbf{x}^*, \mathbf{y}^*$ (Kakutani).*

The last step is:

Lemma 46 *The fixed point is a competitive equilibrium.*

Proof. We already showed that producer maximize profits over the available technology (7) and consumers maximize utility over certain budget set (8). So we only have to show that $M_j(\mathbf{p}^*, \mathbf{x}_j^*) = \mathbf{p}^* \mathbf{x}_j^*$. Suppose $M_j(\mathbf{p}^*, \mathbf{x}_j^*) < \mathbf{p}^* \mathbf{x}_j^*$ some j . Since by Walras law there is another consumer, say k such that $M_k(\mathbf{p}^*, \mathbf{x}_k^*) > \mathbf{p}^* \mathbf{x}_k^*$, the maximization in (9) implies that $\alpha_j^* = 0$. So now (4) implies that this agent is given her minimum consumption which contradicts $M_j(\mathbf{p}^*, \mathbf{x}_j^*) < \mathbf{p}^* \mathbf{x}_j^*$. ■

Thus we have proved;

Theorem 47 *Under A.1-2-3-4', a competitive equilibrium exists.*

5.5 Equilibrium when production is commonly owned

The economy is a special case of Arrow-Debreu with two goods only: corn (good 1) and labor (good 2). To easy notation let the supply of labor of consumer be denoted as $l_j = \omega_{j2} - x_{j2}$ and the consumption of corn by j be denoted by x_j . Let $l = \sum_{j=1}^h l_j$ and $x = \sum_{j=1}^h x_j$. $\mathbf{l}$ and $\mathbf{x}$ are the vectors of labor and consumption. An allocation is a $(\mathbf{l}, \mathbf{x})$.

The definition of a Pareto Efficient allocation is the same as before. Wrt assumptions, on top of those made in the previous sections we assume that the technology is representable by a differentiable, concave production function

$$x = f(l).$$

A Sharing Rule (SR) is a function $\mathbf{S} = (S_1, S_2, \dots, S_h)$ that specifies consumptions of corn as function of labor. Formally,

Definition 48 A sharing rule is a function $\mathbf{S} = (S_1, S_2, \dots, S_h)$ such that

i) $S_j(\mathbf{l}) \geq 0$ for all $j = 1, 2, \dots, h$

$\sum_{j=1}^h S_j(\mathbf{l}) = f(l)$ for all $\mathbf{l} \geq \mathbf{0}$.

Example 49 Proportional

$$S_j(\mathbf{l}) = \frac{l_j}{\sum_{k=1}^h l_k} f(l),$$

egalitarian

$$S_j(\mathbf{l}) = \frac{f(l)}{n}$$

or any convex combination between the proportional and the egalitarian.

Are there sharing rules that are compatible with Pareto efficiency? Formally:

Question: Which sharing rules yield an allocation $(\mathbf{l}, \mathbf{x})$ such that

- a) $\mathbf{x} = \mathbf{S}(\mathbf{l})$
- b) $(\mathbf{l}, \mathbf{x})$ is Pareto efficient for all economies (as defined before) satisfying the conditions in the previous sections.

It was known that the proportional or the equal share SR satisfy a) and b). But Corchón-Puy (*SCW*, 2002) proved a more general result.

Proposition 50 *Any continuous sharing rule satisfies a) and b).*

Problem: Make the proof.

Proof. Identical to the previous one but instead of penalizing those agents whose expenses are larger than their incomes we penalize consumers who consume more than the quantity allocated to them by the sharing rule. ■

Next year: My paper with Iñigo.

6 Extensions: Uncertainty. Time. Non convexities. Incomplete and intransitive preferences.

6.1 Uncertainty

6.1.1 Goods and markets

Recall: m goods (subindex i), h consumers (subindex j), n firms (subindex f).

Vector of excess demands $\mathbf{z}$, Vector of prices $\mathbf{p}$, Vector notation $>>, >, \geq$

Key concept:

Definition 51 *A state of the world is a complete description of the environment.*

Let S be the set of states of the world with $\#S = s$ (finite).

It is common for all agents.

We assume that technology, endowments and preferences depend on s .

Sometimes we will assume that there is a probability distribution on S .

$$\pi = (\pi_1, \pi_2, \dots, \pi_s)$$

π is not necessarily common among agents.

Definition 52 *A good contingent to state k is a good which is only delivered if state k occurs.*

Markets are complete (all goods have a market). Strong assumption.

Markets open before uncertainty is resolved.

Markets balance for every state of the world.

Let $\mathbf{p}_k$ be the vector of prices if state k occurs.

The vector of prices is now $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_s)$.

Deliveries are contingent. Payments are not

6.1.2 Consumers

Definition 53 A state contingent endowment for j is a vector w_j

$$w_j = (w_{j11}, w_{j12}, \dots, w_{j1s}, w_{j21}, w_{j22}, \dots, w_{j2s}, \dots, w_{jm1}, w_{jm2}, \dots, w_{jms})$$

Example 54 One good, peaches. Two states of the world, nice (in which case there are 200 Kg of peaches) and hailstorm. If the hailstorm destroys 100 kg. of peaches, $w_j = (200, 100)$,

Definition 55 A state contingent consumption plan for j is a vector x_j

$$x_j = (x_{j11}, x_{j12}, \dots, x_{j1s}, x_{j21}, x_{j22}, \dots, x_{j2s}, \dots, x_{jm1}, x_{jm2}, \dots, x_{jms})$$

being x_{jik} being consumption of j of good i if state k occurs.

Example 56 Two goods cocido and gazpacho. Two states cold and warm weather.

$$x_j = (x_{j11}, x_{j12}, x_{j21}, x_{j22}).$$

Definition 57 Preferences are defined on x_j .

Example 58 In the previous example if the state is cold the utility function of consumer j is $u_{j1} = (x_{j11})^{.5}$. If the state is warm the utility function of consumer j is $u_{j2} = -x_{j12} + (x_{j22})^{.5}$. If π is the probability that the state is cold the expected utility of consumer j is $u_j = \pi(x_{j11})^{.5} + (1 - \pi)(-x_{j12} + (x_{j22})^{.5})$

It might be that some consumption goods are independent of k .

Remark 59 Preferences are always *ex ante* since the consumer has to specify her demands before uncertainty is resolved.

Remark 60 Consumers bear risk. If in the previous example the consumer buys $x_j = (1, 0, 0, 1)$ and the state cold occurs she does not obtain any gazpacho. If the state is warm she does not obtain any cocido.

Remark 61 Under symmetry wrt states, convex preferences implies risk aversion (A-H, p. 124).

6.1.3 Producers

A state contingent production plan for f is a vector y_f

$$\mathbf{y}_f = (y_{f11}, y_{f12}, \dots, y_{f1s}, y_{f21}, y_{f22}, \dots, y_{f2s}, \dots, y_{fm1}, y_{fm2}, \dots, y_{fms}) \text{ or}$$

$$\mathbf{y}_f = (\mathbf{y}_{f1}, \mathbf{y}_{f2}, \dots, \mathbf{y}_{fs}) \text{ where } \mathbf{y}_{fk} = (y_{f1k}, y_{f2k}, \dots, y_{fmk})$$

being y_{fik} being production of f of good i if state k occurs.

The production possibility set Y_f contains all technologically feasible state contingent plans.

Two possibilities:

1. Production is instantaneous and occurs immediately after the uncertainty is resolved.

Unrealistic but avoids ex post bankruptcies

2. Production takes place before uncertainty.

Realistic but may yield ex post bankruptcies

In the second case feasibility becomes harder to obtain since firms must make investments that only pay off in the relevant state occurs (Mas-Colell example 19.B2) $\rightarrow$ Bankruptcy ex posts.

Let $\mathbf{y}_{fk}$ be the vector input output in state k .

Firms maximize $\sum_{k=1}^s \mathbf{p}_k \mathbf{y}_{fk}$. Firms act as risk neutral.

All risk bearing activities are carried on by households.

It might be that some components of $\mathbf{y}_f$ are independent of k .

6.1.4 Equilibrium

Under the assumptions made in Lectures 3-4-5 a perfectly competitive equilibrium exists and it is Pareto efficient.

This equilibrium is completely silent about ex post bankruptcies.

6.2 Time

The approach is totally identical to the one explained in the case of uncertainty.

Let T be the finite set of periods. $\#T = t$.

We assume that technology, endowments and preferences depend on t .

Markets are complete. All goods have a market. Strong assumption.

Under the assumptions spelled out in Lectures 3-5, an equilibrium exists and it is Pareto efficient.

6.2.1 Extension: Arrow securities

Suppose there are no contingent future markets.

It suffices that one good can be traded at all future dates and states of nature, see Mas-Colell 19D for details.

6.2.2 Non Convexities

As a first approximation they might prevent the existence of equilibrium.

But numbers have a convexifying effect! (Starr (1969)).

Example 62 *1 firm, 2 goods. U-shaped cost curve. An equilibrium might not exist. But now introduce many other firms with different efficient scales. Intuition suggests that when the number of firms is large an approximate equilibrium exists and when the number of firms tend to infinite an equilibrium exists.*

The conditions for the convexifying effect are threefold:

1. Many agents
2. Dispersed characteristics
3. Bounded Production sets.

Make counterexamples

Another approach: Finite economies.

6.3 Incomplete and/or intransitive preferences

How to represent preferences?

$\varphi_i : X_i \rightrightarrows X_i$ with the typical properties. Convex valued with an open non empty graph.

Make examples.

How to define a Walrasian equilibrium?

As a feasible allocation and some prices such that $\forall j$

$$x'_i \in \varphi_i(x_i^*) \rightarrow px'_i > px_i^*$$

All the theorems we have seen can be proved with this approach.

For instance, first fundamental theorem. Problem

7 Structural properties of excess demand functions. Uniqueness and stability of competitive equilibrium.

7.1 Properties of excess demand functions

Recall: m goods (subindex i), h consumers (subindex j), n firms (subindex f).

Vector of excess demands $\mathbf{z}$, Vector of prices $\mathbf{p}$, Vector notation $>>, >, \geq$

Motivation: The shape of excess demand functions determines:

The number of equilibria

How equilibrium changes with the parameters of the economy.

Stability of equilibria.

Make a picture.

Properties of excess demand functions, $F : S \rightarrow \mathbb{R}^m$:

1. **C.** Continuous and well defined functions in the interior of S
2. **W.** Walras law.
3. **B.** If undefined at $\mathbf{p}^0$, $\lim_{\mathbf{p} \rightarrow \mathbf{p}^0} \sum_{i=1}^m F_i(\mathbf{p}) \rightarrow \infty$ (Arrow-Hahn pp. 102-3).

What else?

7.1.1 The "good", particular, case

In some cases (excess) demand functions have additional properties to C-W-B above as in the following case, discovered by Antonelli (1886) and rediscovered by Gorman and Nataf in 1953.

Consumer demand is a function $D_j : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^m$, $j = 1, 2, \dots, h$.

It is derived from utility maximization.

Market demand is a function $D : S \times \mathbb{R}_+^h \rightarrow \mathbb{R}_+^m$

It is derived from the aggregation of individual demands $D = \sum_{j=1}^h D_j$.

Definition 63 *A preference relation is homothetic if it is representable by a utility function which is homogeneous of degree one.*

Individual demand functions derived from homothetic preferences look like $D_j = M_j F_j(p)$.

Theorem 64 *Market demand is a consumer demand function iff all consumers are identical and homothetic.*

Proof. If all preferences are identical and homothetic each individual demand function is $M_j F(p)$ so market demand is $\sum_{j=1}^h M_j F(p) = F(p) \sum_{j=1}^h M_j = F(p)M$ where $M = \sum_{j=1}^h M_j$.

The converse follow from giving all the income to, say consumer j in which case $D = D_j$. Repeating this argument for all consumers we have that all individual demand functions must be identical. Linearity on M_j follows from $D(p, M_1 + M_2) = D(p, M_1) + D(p, M_2)$ for all $M_1, M_2 \geq 0$. ■

An identical proof follows for excess demand functions replacing income by endowments.

Theorem 65 *Excess market demand is a consumer excess demand function iff all consumers are identical and homothetic.*

7.1.2 The "bad", general case

The result was conjectured and proved in a special case by Sonnenschein (1973), proved in its generality by Mantel (1974) and refined by Debreu (1974).

Definition 66 *A classical exchange economy is one in which all consumers have continuous, strictly convex and monotone preferences.*

The truncated simplex $S_\lambda = \{\mathbf{p} \in S \mid \mathbf{p} \geq e\lambda\}$ (By **B**).

Theorem 67 *Let F be a continuous excess demand function. For any $\lambda > 0$ there is a classical exchange economy with $m = h$ which yields F as an aggregate excess demand function on S_λ .*

Geometric proof for $m = h = 2$ (Edgeworth box).

Step 1. Draw the box with endowments $(2, 0)(0, 2)$.

Single out a line $L = (1, 0)(2, 1)$ where most of the action is going to be.

Step 2. Draw homothetic preferences such that the utility maximizing choices of consumer 1 are in L .

Notice that sufficiently smooth perturbations do not destroy homotheticity.

Step 3. Bring a symmetric consumer.

All the points in L are Walrasian equilibria.

Step 4. Go back to F . Make a smooth perturbation in the preferences of agent 1 such that the excess demand for good 1 is $\varepsilon F_1(p_1)$ for ε sufficiently small. Multiply endowments by $1/\varepsilon$ and, by homothetic preferences, excess demand multiplies by $1/\varepsilon$ so we are done.

7.1.3 Extensions

Mantel (*JET* 1976) showed that the theorem holds even if agents had homothetic preferences (and arbitrary endowments).

Kirman and Koch (*RES* 1986) showed that the theorem holds even if agents are identical and have collinear endowments.

7.1.4 Scape Routes

Becker (*JPolEcon* 1962). If individual demands satisfy Walras's Law and are sufficiently dispersed, then market demand will look approximately Cobb-Douglas. For example, suppose a continuum of consumers and that demands are distributed uniformly over the budget line. Then the average demand will be $(M/2p_1, M/2p_2)$ which is the demand of a Cobb-Douglas consumer with income M and exponents .5, .5. It does not require individual demand to be derived from utility maximization.

Hildenbrandt (*Econometrica* 1983). If individuals have identical tastes and the distribution of income is non increasing in $[0, \bar{M}]$ the market demand obeys the Law of Demand $\Delta p \Delta x < 0$, $\Delta p \neq 0 \rightarrow$ uniqueness.

7.1.5 References

The formal results are taken from

Shafer, W. and H. Sonnenschein "Market Demand and Excess Demand Functions" Theorems 1-2 and the geometric postscript to Section 4. Chp. 14 in Handbook of Mathematical Economics, K. J. Arrow and M.D. Intriligator (eds). Elsevier

The implications of the results obtained here are in:

Kirman, A. "Whom or what does the representative individual represent?" *JEPersp.* 1992.

7.2 Uniqueness

Importance of uniqueness for comparative statics and stability.

Make a picture.

We have to dispose of one good (but see GS later on).

If not there is a continuum of equilibria

So for in this section, unless stated contrarily, we consider only $m - 1$ goods.

Let J be the $(m - 1) \times (m - 1)$ Jacobian of excess supply $\mathbf{s} = -\mathbf{z}$.

Sometimes excess supply are handier than excess demands

Typical element $s_{ij} \equiv -\frac{\partial z_i(\mathbf{p})}{\partial p_j}$

In order to get rid of useless technicalities we will assume that any equilibrium price vector is strictly positive.

We have two approaches to the question of uniqueness of equilibrium.

Classical Approach: Putting restrictions (unwarranted) on excess demands.

Gale-Nikaido property.

Special cases: DD, GS, WARP

We obtain a unique equilibrium

Modern Approach: We only care for finiteness because this allows for local comparative statics.

Sard Theorem.

We obtain a finite number of equilibria "almost everywhere".

7.2.1 Classical approach

Implicit function theorem assures local uniqueness.

Global uniqueness is not a trivial business.

Samuelson made a conjecture that turned out to be wrong.

The following result is an easy one.

Theorem 68 *Suppose there is a unique consumer with a strictly quasi concave utility function and production sets are convex. Then there is a unique allocation which is a Walrasian equilibrium. If the utility function is differentiable the equilibrium vector of prices is unique too.*

Proof. Since Walrasian equilibrium is Pareto efficient if there are two allocations that are a Walrasian equilibrium there must yield the same utility. But taking a convex combination of these allocations, this combination is feasible and yields more utility than the Walrasian equilibria which contradicts our starting point. ■

The assumptions of our Theorem 6 are slightly less general than WARP.

$$\mathbf{p}\mathbf{z} < \mathbf{p}\mathbf{z}' \text{ or } \mathbf{p}'\mathbf{z}' < \mathbf{p}'\mathbf{z} \text{ for all } \mathbf{p}, \mathbf{p}', \mathbf{z} = \mathbf{z}(\mathbf{p}) \text{ and } \mathbf{z}' = \mathbf{z}(\mathbf{p}')$$

Theorem 69 *WARP implies a unique equilibrium price vector.*

Proof. See Takayama pp.280-1 or Mas-Colell pp. 608-9 (weaker) ■

In the same vain we can prove that equilibrium prices are unique in several special cases, e.g. when the economy is Leontieff or when there is no trade (see Mas-Colell pp. 614-5).

What about more general results?

The answer was given by Gale and Nikaido (1965 *Mathematische Annalen*)

A real matrix J is a P-matrix if all its principal minors are positive.

In economics a P-matrix is called Hicksian or the Gale-Nikaido property.

Theorem 70 (*Gale-Nikaido*) *If J is a P-matrix defined in a closed rectangular region of $\mathbb{R}^{m-1}$, there is a unique $\mathbf{p}$ such that $\mathbf{z}(\mathbf{p}) = \mathbf{0}$.*

What conditions ensure that J is a P-matrix? We present two.

Definition 71 *J has Dominant Diagonal (DD) if there are $m - 1$ numbers $d_i > 0$ such that*

$$|s_{ii}| d_i > \sum_{k \neq i} |s_{ik}| d_k, \quad i = 1, 2, \dots, m - 1.$$

Lemma 72 *If a DD matrix has positive diagonal entries it is a P-matrix*

Proof. See, e.g. Arrow and Hahn pp. 234-5. ■

Corollary 73 *If J has DD and positive diagonal entries equilibrium prices are unique.*

DD has a kind of Marshallian partial equilibrium flavour.

Positive diagonal entries forbids Giffen goods.

Another special case of a P-matrix is Gross Substitutes.

Definition 74 *Goods are Gross Substitutes (GS) if*

$$\forall i, k = 1, 2, \dots, m, \quad i \neq k, \quad s_{ik} < 0.$$

The definition can be given without differentiability

Example: Cobb-Douglas preferences

Theorem 75 *If all goods are GS equilibrium prices are unique.*

Proof. Let $\mathbf{p}$ be an arbitrary price vector. Let

$$v_i = \frac{p_i}{p_i^*}.$$

Let $v_k = \max_{i=1,2,\dots,m-1} v_i = m$. By Homogeneity of degree zero, $s_k(\mathbf{p}^*) = s_k(m\mathbf{p}^*)$. Now construct a sequence of prices starting at $m\mathbf{p}^*$ and ending at $\mathbf{p}$ such that at each step p_k remains unchanged while at least some other price is lowered and none is raised. By GS, $s_k(\mathbf{p}) > 0$ so equilibrium price is unique. ■

7.2.2 Modern approach

It follows from Sard's lemma.

k -continuously differentiable functions are denoted by $\mathbb{C}^k$

$\mathbb{C}^1$ functions are defined on open subsets of $\mathbb{R}^t$.

A critical point of a $\mathbb{C}^1$ function F is one for which J vanishes.

A critical value of a $\mathbb{C}^1$ function F is $F(x)$ where x is a critical point

Theorem 76 *The set of critical values of a $\mathbb{C}^1$ function $F : U \rightarrow \mathbb{R}^t$ where $U \subset \mathbb{R}^t$ has Lebesgue measure zero.*

Make a picture.

Now we are prepared to our main result here.

Theorem 77 *If demand functions are $\mathbb{C}^1$ and B holds the set of initial endowments for which there are an infinite number of price equilibrium vectors is a closed set of Lebesgue measure zero.*

Proof. Define $F : U \rightarrow \mathbb{R}^{mh}$ where $U = [intS \times (0, \infty)] \times \mathbb{R}^{h-1}$ as follows

$$\begin{aligned} F_1 &= f_1(p, y_1) + \sum (f_t(p, p \cdot w_t) - w_t) \\ F_t &= w_t \quad t = 2, 3, \dots, h \end{aligned}$$

Firstly realize that the domain and the range have the same dimensionality, namely mh .

Next, if $\mathbf{p}$ is an equilibrium price vector and $y_1 = p w_1$ then $F = w$. Now Sard's

lemma guarantees that almost all w are not critical values. Let w be a regular value of F .

Secondly we show that finitely many p such that are equilibria. Suppose not, so there are an infinite number of such p 's. There is a limit value call it p_∞ . This cannot be in the boundary of S by B. And if it converges to a point interior to S , w is a critical point which contradicts our initial assumption.

Finally the closedness follows from considering a sequence of critical points and from the continuity of J . ■

8 Public Goods. Mechanism design.

Summary 78 ◦ *Definition of a public good. Economies with public goods*

- *Efficiency. FOC. Kolm triangle*
- *Lindahl equilibrium. Efficiency, existence, criticism*
- *Mechanisms for resource allocation*
- *Dominant strategy implementation. Revelation principle. Impossibility results*

8.1 Allocation of Public Goods

m private goods (sub. i), l public goods (sub. k), h consumers (sub. j), 1 firm.

Vector of prices of private goods $\mathbf{p}$. Vector of prices of public goods $\mathbf{q}$.

Vector of public goods $\mathbf{y}$. Vector of private inputs $\mathbf{z}$ POSITIVE.

A public good is one whose consumption is public.

Example 79 *Army, Police, Roads, Trains, Justice, Lectures*

streetlights, sewer systems, bridges, parks

TV, radio, inventions, pollution

No congestion.

Not necessarily good.

It is not necessarily offered by the public sector.

Make a picture

A consumption bundle for consumer j is a $(\mathbf{x}_j, \mathbf{y}) \in X_j \subset \mathbb{R}^{m+l}$.

Preferences for consumer i are represented by a utility function $u_i(\mathbf{x}_i, \mathbf{y})$.

Technology is represented by a set $Y \subset \mathbb{R}^{m+l}$ or a function $T(\mathbf{z}, \mathbf{y}) = 0$

An allocation is a $(\mathbf{x}, \mathbf{y}) \in \times_{j=1}^h X_j$.

Definition 80 *An allocation $(\mathbf{x}, \mathbf{y})$ is feasible if $\exists \mathbf{z}$ such that $(\mathbf{z}, \mathbf{y}) \in Y$ and for all $i = 1, 2, \dots, m$,*

$$\sum_{j=1}^h (x_{ji} - w_{ji}) + z_i \leq 0.$$

8.2 Efficiency

Definition 81 A feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no feasible allocation $(\mathbf{x}', \mathbf{y}')$ such that $u_j(\mathbf{x}'_j, \mathbf{y}') > u_j(\mathbf{x}_j, \mathbf{y})$ for all $j = 1, 2, \dots, h$.

Assume

Utility functions are continuous, non satiated, concave

Production set is closed and convex.

Then we have the following:

Lemma 82 An allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient iff maximizes a weighted sum of utility functions over the feasible set, i.e. $\exists \alpha \in S$ such that $(\mathbf{x}, \mathbf{y})$ maximizes

$$\sum_{j=1}^h \alpha_j u_j(\mathbf{x}_j, \mathbf{y}) \text{ subject to } \sum_{j=1}^h (x_{ji} - w_{ji}) \leq -z_i, \forall i = 1, 2, \dots, m \text{ and } T(\mathbf{x}, \mathbf{y}) = 0.$$

Let

$$L = \sum_{j=1}^h \alpha_j u_j(\mathbf{x}_j, \mathbf{y}) - \beta(T(\mathbf{x}, \mathbf{y})) - \sum_{i=1}^m \gamma_i \left(\sum_{j=1}^h (x_{ji} - w_{ji}) + z_i \right)$$

Then, FOC of this maximization. Assuming interiority are

$$\sum_{j=1}^h \alpha_j \frac{\partial u_j(\mathbf{x}_j, \mathbf{y})}{\partial y_k} - \beta \frac{\partial T(\mathbf{x}_j, \mathbf{y})}{\partial y_k} = 0, k = 1, 2, \dots, l$$

$$\alpha_j \frac{\partial u_j(\mathbf{x}_j, \mathbf{y})}{\partial x_{ji}} - \gamma_j = 0, j = 1, 2, \dots, h, i = 1, 2, \dots, m.$$

$$\beta \frac{\partial T(\mathbf{x}_j, \mathbf{y})}{\partial z_i} + \gamma_i = 0, i = 1, 2, \dots, m.$$

From these equations we obtain that

$$\sum_{j=1}^h \alpha_j \frac{\partial u_j(\mathbf{x}_j, \mathbf{y})}{\partial y_k} + \gamma_i \frac{\frac{\partial T(\mathbf{x}_j, \mathbf{y})}{\partial y_k}}{\frac{\partial T(\mathbf{x}_j, \mathbf{y})}{\partial z_i}} = 0, \quad i = 1, 2, \dots, m, \quad k = 1, 2, \dots, l.$$

$$\alpha_j \frac{\partial u_j(\mathbf{x}_j, \mathbf{y})}{\partial x_{ji}} = \alpha_s \frac{\partial u_s(\mathbf{x}_j, \mathbf{y})}{\partial x_{si}} = \gamma_j, \quad j, s = 1, 2, \dots, h, \quad i = 1, 2, \dots, m.$$

$$\sum_{j=1}^h \alpha_j \frac{\partial u_j(\mathbf{x}_j, \mathbf{y})}{\partial y_k} + \alpha_s \frac{\partial u_s(\mathbf{x}_j, \mathbf{y})}{\partial x_{si}} \frac{\frac{\partial T(\mathbf{x}_j, \mathbf{y})}{\partial y_k}}{\frac{\partial T(\mathbf{x}_j, \mathbf{y})}{\partial z_i}} = 0, \quad s = 1, 2, \dots, h.$$

Thus we finally obtain that

$$\sum_{j=1}^h \frac{\frac{\partial u_j(\mathbf{x}_j, \mathbf{y})}{\partial y_k}}{\frac{\partial u_j(\mathbf{x}_j, \mathbf{y})}{\partial x_{ji}}} = - \frac{\frac{\partial T(\mathbf{x}_j, \mathbf{y})}{\partial y_k}}{\frac{\partial T(\mathbf{x}_j, \mathbf{y})}{\partial z_i}}, \quad k = 1, 2, \dots, l, \quad i = 1, 2, \dots, m.$$

The sum of marginal relations of substitution between any pair of public and private goods = marginal rate of transformation between these two goods (Lindahl, Bowen, Samuelson).

Difference with the private good case.

Graphically: Kolm triangle. $i = l = 1, h = 2$.

Constant returns. $y = z$.

Feasibility: $x_1 + x_2 + z = w_1 + w_2 \Leftrightarrow x_1 + x_2 + y = w_1 + w_2$

Construction;

Draw preferences. Consumer 2 with the x axis to the left.

Signal endowments.

Change the angle between axis to 60° .

Move one against the other.

Stop when endowments intersect.

Property: The sum of all distances is constant.

→ Any point in the triangle is feasible

Identify Efficient allocations.

8.3 Lindahl equilibrium

8.3.1 Definition

Properties of a satisfactory allocation mechanism for public goods:

1. The decision about private and public goods should be decentralized.
2. The decision about public goods should be unanimous and efficient.

Lindahl as formalized by Samuelson (1954).

Make a picture in the case $i = l = 1$, $h = 2$.

Personalized prices.

Definition 83 $(\mathbf{x}^L, \mathbf{y}^L, \mathbf{z}^L)$ is a Lindahl equilibrium if $(\mathbf{x}^L, \mathbf{y}^L)$ is feasible (corresponding to $\mathbf{z}^L$) and there is a price vector $(\mathbf{p}^L, \mathbf{q}^L) \in \mathbb{R}^{m+lh}$ such that

- a) For all $j = 1, 2, \dots, h$, $\mathbf{x}_j^L$ maximizes $u_j(\mathbf{x}_j, \mathbf{y})$ over $\mathbf{q}_j^L \mathbf{y} + \mathbf{p}^L \mathbf{x}_j \leq \mathbf{p}^L \mathbf{w}_j + \theta_j (\sum_{s=1}^h \mathbf{q}_s^L \mathbf{y}^L - \mathbf{p}^L \mathbf{z}^L)$
- b) $(\mathbf{y}^L, \mathbf{z}^L)$ maximizes $\sum_{s=1}^h \mathbf{q}_s^L \mathbf{y} - \mathbf{p}^L \mathbf{z}$ over $(\mathbf{y}, \mathbf{z}) \in Y$.

We don't assume that personalized prices are positive.

Extensions Kaneko, Mas-Colell-Silvestre.

8.3.2 Properties of Lindahl equilibrium

Efficiency.

Proposition 84 *If consumers are locally non satiated, Lindahl equilibrium is efficient*

Proof. Suppose not. So there is another feasible allocation $(\mathbf{x}', \mathbf{y}', \mathbf{z}')$ such that $u_j(\mathbf{x}'_j, \mathbf{y}') > u_j(\mathbf{x}^L_j, \mathbf{y}^L)$ for all j . This implies that

$$\mathbf{q}^L_j \mathbf{y}' + \mathbf{p}^L \mathbf{x}'_j > \mathbf{q}^L_j \mathbf{y}^L + \mathbf{p}^L \mathbf{x}^L_j = \mathbf{p}^L \mathbf{w}_j + \theta_j \left(\sum_{s=1}^h \mathbf{q}^L_s \mathbf{y}^L - \mathbf{p}^L \mathbf{z}^L \right).$$

Adding over all consumers we obtain that

$$\sum_{s=1}^h (\mathbf{q}^L_s \mathbf{y}' + \mathbf{p}^L \mathbf{x}'_s - \mathbf{p}^L \mathbf{w}_s) > \sum_{s=1}^h \mathbf{q}^L_s \mathbf{y}^L - \mathbf{p}^L \mathbf{z}^L$$

Now by profit maximization,

$$\sum_{s=1}^h \mathbf{q}^L_s \mathbf{y}^L - \mathbf{p}^L \mathbf{z}^L \geq \sum_{s=1}^h \mathbf{q}^L_s \mathbf{y}' - \mathbf{p}^L \mathbf{z}'$$

and so

$$\begin{aligned} \sum_{s=1}^h (\mathbf{q}^L_s \mathbf{y}' + \mathbf{p}^L \mathbf{x}'_s - \mathbf{p}^L \mathbf{w}_s) &> \sum_{s=1}^h \mathbf{q}^L_s \mathbf{y}' - \mathbf{p}^L \mathbf{z}' \Leftrightarrow \\ \sum_{s=1}^h (\mathbf{p}^L \mathbf{x}'_s - \mathbf{p}^L \mathbf{w}_s) &> -\mathbf{p}^L \mathbf{z}' \end{aligned}$$

which contradicts feasibility. ■

Existence of equilibrium

Convert an economy with public goods into an economy with private goods.

Let $\mathbf{y}_j$ be the vector of public goods for j .

Introduce in Y the restriction $\mathbf{y}_1 = \mathbf{y}_2 = \dots, \mathbf{y}_h$ (perfect complementarity).

Make all other assumptions (convexity, continuity...).

But..... THE CHEAPER POINT ASSUMPTION

8.3.3 Criticism: Incentives

Samuelson (1954): "Each person will be interested in giving false signals"

In other words, monopsony agents are not price takers.

Formalize this by using the Kolm triangle

As Hurwicz noticed this is a completely general point:

It applies to any kind of environments

Private goods

Social choice

Public goods

To any solution which

A) Is defined in a large domain of preferences

B) Satisfies individual rationality (in 1&2 above) or non dictatorship

C) Pareto Efficiency

And the theorem says

"In environment 1/2/3 no SCC satisfying A-B-C is incentive compatible in DS"

8.3.4 Impossibility of implementing interesting SCC in DS

We assume (Beviá-Corchón, ET 1995):

$m = 1$, which is the numeraire.

constant returns to scale. Convex cost function $c(\mathbf{y})$.

Utility functions are differentiable, concave and quasi-linear in the private good $u_j = v_j(\mathbf{y}) + x_j$

Definition 85 A feasible allocation $(\mathbf{x}^{Ms}, \mathbf{y}^{Ms})$ is the monopoly point for agent s if there is a $\mathbf{q}^{Ms}$ such that

- a) For all $j \neq s$ $(\mathbf{x}_j^{Ms}, \mathbf{y}^{Ms})$ maximizes $u_j(\mathbf{x}_j, \mathbf{y})$ over $\mathbf{q}_j^{Ms} \mathbf{y} + x_j \leq w_j$
- b) $\mathbf{y}^{Ms}$ maximizes $\sum_{s=1}^h \mathbf{q}_s^{Ms} \mathbf{y} - c(\mathbf{y})$
- c) $(\mathbf{x}^{Ms}, \mathbf{y}^{Ms})$ maximizes $u_s(\mathbf{x}_s, \mathbf{y})$ over the set of feasible allocations fulfilling a) and b).

Make a picture in the Kolm triangle.

Let u_j^T , $T = L, M, j$ be the utility of consumer j in Lindahl and monopoly point.

Let e to be an economy

$$e = (u_1, u_2, \dots, u_h)$$

$$u_i \in U_i. \text{ Let } U = \times_{j=1}^h U_j.$$

$$u_i : A \rightarrow \mathbb{R}.$$

Definition 86 A social choice correspondence is a mapping $\phi : U \rightrightarrows A$.

This mapping embodies social objectives.

Examples: Lindahl, Walras, Envy-Free

We will assume in our result that it is a function (SCF).

Define incentive compatibility.

We now make our domain assumption

A' The domain of ϕ includes all linear economies.

And we make the following definition

Definition 87 *A regular economy is one in which all utility functions are strictly concave, Lindahl and monopoly point are interior and there is a s such that $u_s^{Ms} > u_s^L$.*

Theorem 88 *No SCF satisfying A'-B-C and picking up interior points is incentive compatible at any regular economy.*

Corollary 89 *No ϕ satisfying A'-B-C and picking up interior points is implementable in DS.*

Proof. Step 1. Identify in the Kolm triangle the set of efficient and individually rational allocations for u . $\phi(u)$ must be contained in this set.

Step 2. Any j by sending linear preferences with slope identical to Lindahl prices, i.e. $u_j(\mathbf{x}_j, \mathbf{y}) = \mathbf{q}_j^L \mathbf{y} + x_j$ can achieve, at least u_j^L .

Step 3. Individual s can now send linear preferences with slope identical to Monopoly prices, i.e. $u_s(\mathbf{x}_s, \mathbf{y}) = \mathbf{q}_s^{Ms} \mathbf{y} + x_s$ and achieve, at least u_s^{Ms} . ■

8.4 References:

Corchon, L. "The Theory of Implementation of Socially optimal Decisions in Economics". McMillan, 1996.

9 Existence of social welfare functions.

9.1 Motivation

Can we interpret the weighted sum of utilities as a SWF?

Weights are chosen to balance the market.

Very particular set up (the market).

Very particular SWF (weighted utilitarian).

Can we aggregate individual preferences into a social preference?

Representative citizen, consumer, etc.

Ubiquitous in Macro, IO, Public Economics, Welfare economics, etc.

Let us cast the problem in a formal way.

9.1.1 The model

Let A be the finite set of feasible allocations

We will call the elements of A alternatives.

Let $\succsim_j$ be a preference relation for j .

It is assumed reflexive, transitive and complete.

Let $\succ_j$ be the strict preference relation derived from $\succsim_j$

Let $\mathfrak{R}_j$ be the set of all possible preferences of j .

Let $(\succsim_1, \succsim_2, \dots, \succsim_n)$ be denoted by $\succsim$.

Let $\mathfrak{R}$ be the set of all conceivable preferences.

A social welfare functional is a function $S : \times_{j=1}^n \mathfrak{R}_j \rightarrow \mathfrak{R}$

We denote an element in the range of S as $\succsim_S$.

Remark 90 S is not a constitution (which is a very complicated object).

S is not a voting scheme which only selects one alternative (the domain is A).

Only rankings count, intensity of preferences does not.

9.1.2 Preference aggregation with two alternatives

$\#A = 2$.

Why $\#A = 2$? May's theorem: Majority voting.

S is **anonymous** if any permutation of the preferences across agents does not matter.

S is **neutral** if the social preference is reversed when we reverse the preferences of all agents.

S is **positive responsive** when $x \succsim_S y$ and x raises in the ranking of an agent, $x \succ_S y$.

Theorem 91 *S satisfies anonymity, neutrality and is positive responsive iff it is the majority voting*

Proof. Necessity is obvious, for sufficiency see Mas-Colell p. 792. ■

When there is a tie, majority voting SWF yields indifference.

Remark 92 *When we have "money" majority voting does not necessarily yield PE alternatives*

Remark 93 *The implementation of majority voting is troublesome. A situation in which all agents vote for the worst alternative is a Nash equilibrium.*

9.1.3 Preference aggregation for three or more alternatives

Unrestricted Domain (U): The domain of S is $\mathfrak{R}^n$

Motivation: All opinions are admissible.

Avoids trivial domains with no conflict.

Pareto Efficiency (P): If $x \succ_j y \forall j$, $x \succ y$.

Motivation: Social preferences must be responsive to individual preferences.

Social preferences reflect unanimous voting over any pair of alternatives.

Non Dictatorship (D): $\nexists j$ such that $S(\succ) = \succ_j$ for all $\succ_j \in \mathfrak{R}_j$, $\succ \in \mathfrak{R}$.

Motivation: Obvious in static models. Not obvious in Dynamic models.

There are many social aggregation procedures that satisfy **U**, **P** and **D**.

1. Select an individual h and translate the preferences of h to social preferences except that we flip two alternatives.
2. Put the alternatives which can be Pareto ranked on top of the social ranking and write the others in a fixed order.
3. Make the social ranking to be the ranking of someone except for one profile of preferences.

So in order to discard silly proposals we make our last assumption

Independence of Irrelevant Alternatives (I): The social ranking of any two alternatives depends only on individual rankings of these two alternatives.

Not confuse with Nash IIA.

It is the only assumption that can be challenged.

It is not on the same level as the others.

It eliminates things which are widely used:

Plurality rule:

The winner is the candidate who is more voters' favorite candidate (i.e., the candidate more voters rank first) than any other.

It is a weak extension of majority voting to $\#A > 2$.

Plurality rule is used to elect Senators and Representatives in the U.S. and Members of Parliament in Britain (where it's called "first-past-the-post").

Problems of the plurality rule:

A Condorcet winner (CW) is a candidate preferred by a majority to any other candidate

It is a strong extension of Majority voting to $\#A > 2$.

Not always exists (Condorcet paradox).

Example 94 Plurality may select an alternative that is not a CW.

Suppose three candidates, X , Y , and Z and the preferences in the Table below. Y is a Condorcet winner because a majority of voters ($35\% + 25\% = 60\%$) prefer Y to X , and a majority ($40\% + 35\% = 75\%$) prefer Y to Z .

40%	35%	25%
X	Y	Z
Y	Z	Y
Z	X	X

Example 95 Plurality is affected by spoilers. *U.S. 2000 presidential election. The election came down to Florida: if Bush carried the state, he would become president, and the same for Gore. Florida—like most other states—uses plurality rule to determine the winner. Bush got somewhat fewer than six hundred more votes than Gore that gave Bush the presidency. But a third candidate in Florida, Ralph Nader obtained nearly one hundred thousand votes, and it is likely that, had he not been on the ballot, a large majority of these voters would have voted for Gore. That means that Gore would probably have won.*

Nader who got less than 2% of Florida's vote determined the election.

That seems highly undemocratic.

The independence axiom serves to rule out spoilers.

9.1.4 The result

Theorem 96 (*Arrow, 1951*). *There is no social welfare functional fulfilling UPDI*

Proof. (Geanakoplos *ET* 2005, inspired in Barberá *ELett* 1980). The proof has four steps

- 1). *If every agent puts b either at the top or at the bottom of her ranking, society must do as well.* Suppose not so $\exists a, c$ such that $a \succeq_S b \succeq_S c$ with at least one preference strict so $a \succ_S c$. By **I** this would hold if every individual moved c above a (since b is at the extreme this change does not disturb the relationships between a and b and c and b). By **P** $c \succ_S a$ a contradiction.
- 2) *There is a j such that by changing her order at some profile can move b from the bottom to the top of $\succeq_S$.* Suppose b is at the bottom of all rankings. By **P** b is at the bottom of the social ordering. Let individuals successively move b from the bottom to the top of their orderings. Finally b must be at the top of the social ordering. The first voter who causes this change, j , is the required voter. For future reference we will call the list of orderings after the change AC.
- 3) *j is the dictator over any pair not involving b .* Construct a new ordering from AC by letting j move a above b and above c so $a \succ_j b \succ_j c$. Let all other agents to rearrange their rankings of a and c leaving b at the extreme. By **I** $a \succ_S b$ and $b \succ_S c$ so $a \succ_S c$. By **I** $a \succ_S c \Leftrightarrow a \succ_j c$.
- 4) *j is a dictator over any other pair.* Take any other alternative c and put it at the bottom like in step 2). From step 3) there is a voter, who is a dictator by any pair not involving c . But j can affect $\succ_S$ over a and b so both dictators are the same. ■

Extensions: A with an infinite number

9.1.5 Domain restrictions

Meaningful restrictions on the domain produce positive results.

Definition 97 *Preferences are single-peaked if when $A \subseteq \mathbb{R}$ they are quasiconcave.*

Example; Euclidean preferences.

Theorem 98 *The Median Voter Theorem (Black 1948, Downs 1957): With single-peaked rankings, the median voter is a Condorcet winner.*

Proof. Make a picture. ■

Economic domains.

Social welfare functions when preferences are convex, strictly monotonic, and continuous

Ehud Kalai, Eitan Muller, Mark A. Satterthwaite (1979 Public Choice).

If the space of alternatives is $\mathbb{R}_+^n$, where each dimension represents a different public good and if each person's preferences are convex, continuous, and strictly monotonic, then no social welfare function exists that satisfies UPDI.

Maskin (1976 unpub.)

When there are at least two private goods, Weak Pareto and Nondictatorship are inconsistent with a nonnegative responsiveness condition (a strengthening of Independence of Irrelevant Alternatives) on the domain of selfish, continuous, monotonic, and convex preferences if each individual consumption set is the positive orthant.

Other issues:

Abstention:

Sincere voting: Gibard-Satterthwaite. However single peaked preferences provides a domain in which strategy proofness holds.

10 Cournot-Nash cooperative foundations of competitive equilibrium

10.1 Introduction

Consider a toy model (Mickey mouse) model of imperfect competition

There is a finite number of firms n

They compete à la Cournot: strategies are quantities.

Why? The Bertrand paradox

Price making is compatible with Cournot competition:

1. Two stage game: capacity precommitment followed by price competition.
Kreps-Scheinkman: "Quantity precommitment and Bertrand competition yield Cournot outcomes". *Bell J. of Econ*, 14, 1983 (crucial assumption: efficient rationing. Davidson-Deneckere "Long-Run Competition in Capacity, Short-Run Competition in Price, and the Cournot Model" *Rand J. of Econ.* 17, 1986. Other assumptions 2 firms, homogeneous product. Generalized to n firms differentiated products by Friedman: "On the strategic importance of prices vs. quantities". *Rand J. of Econ*, 19 1988.
2. Interpretation of the residual demand (Corchón 1995, 2001. pp.19-20).

Firms face a (piecewise) linear inverse demand curve

$$p = \max(0, A - bx), \quad A, b > 0.$$

p = market price $x = \sum_{i=1}^n x_i$, where x_i = output of firm i .

This function comes from inverting $x = \max(0, A - p)/b$.

Firms are identical with a constant marginal cost of $c < A$.

Profits for firm i $\pi_i = p(x)x_i - cx_i = (p(x) - c)x_i$

Forget for a moment the non-negativity constraints, then

$$\pi_i = (A - bx - c)x_i = (a - bx)x_i \text{ where } a \equiv A - c.$$

FOC of profit maximization are

$$a - bx - bx_i = 0, i = 1, 2, \dots, n.$$

SOC hold. Adding up we obtain that

$$x^* = \frac{na}{(n+1)b}, p^* = \frac{A+nc}{n+1}, x_i^* = \frac{a}{(n+1)b}, i = 1, 2, \dots, n.$$

Show that this is the equilibrium of the game with non negativity constraints and that there are no other equilibria.

$$x^C = \frac{na}{(n+1)b}, p^C = \frac{A+nc}{n+1}, x_i^C = \frac{a}{(n+1)b}, i = 1, 2, \dots, n.$$

Remark 99 *There is a unique Cournot-Nash equilibrium*

Equilibrium is different from a WE where $p^W = c$ and so $x^W = a/b$.

Let us replicate the economy by increasing the number of firms n

Remark 100 *When $n \rightarrow \infty$ total output and price in Cournot-Nash equilibrium converge to the price and aggregate output of the WE*

Difficulty: Individual outputs tend to zero. Firms face a downward sloping demand but they are insignificant and this is what drives the result.

What if we replicate firms and consumers?

$$x = h \frac{A-p}{b}$$

(h = number of consumers). Now the inverse demand function is

$$p = \max\{0, A - \frac{b}{h}x\}. \quad (10)$$

Repeating previous calculations we obtain that

$$x^C = \frac{nah}{(n+1)b}, p^C = \frac{A+nc}{n+1}h, x_i^C = \frac{ah}{(n+1)b}, i = 1, 2, \dots, n.$$

And WE is now $p^W = c$ and $x^W = ah/b$ $x_i^W = ah/bn$.

By multiplying h and n by the same factor we obtain that in the limit

$$x_i^C(\infty) = \frac{a}{b} \text{ and } x_i^W(\infty) = \frac{a}{b}.$$

So now firm are truly price-takers: The inverse demand function tends to A .

In the next section we will try to replicate this finding in a GE framework

10.2 Imperfect competition in general equilibrium: Existence of a Cournot-Nash equilibrium

The outstanding paper is Gabsewicz-Vial "Oligopoly à la Cournot in General equilibrium" *J. Econ Theory* 4, 1972.

Assumptions on consumers are the same as under perfect competition.

They are price-takers.

We will assume $X_j = \mathbb{R}_+^m$ for simplicity.

How do we construct the inverse demand function?

Let $\mathbf{y}$ be given as endowments to consumers.

$$(w_{j1} + \sum_{f=1}^n \theta_{jf} y_{f1}, \dots, w_{jm} + \sum_{f=1}^n \theta_{jf} y_{fm}) = \omega_j(\mathbf{y})$$

Then we can define a Walrasian equilibrium relative to endowments $\omega(\mathbf{y})$.

Problem #1: Some components of $\mathbf{y}$ are negative.

This could rock the boat because $\omega_j(\mathbf{y}) \in \text{int } X_j$.

Remedial assumption #1: $\mathbf{y} \geq \mathbf{0}$. Inputs are non marketable (cooperatives)

$\omega_j(\mathbf{y}) \geq \mathbf{0}$ won't do it because endowments must be very large, so why consumers are price-takers?

With the remedial assumption #1 a WE with endowments $\omega(\mathbf{y})$ exists

Let $\mathbf{p}(\mathbf{y})$ be WE equilibrium prices relative to $\mathbf{y}$ (Walrasian correspondence)

Problem #2: $\mathbf{p}(\mathbf{y})$ is not necessarily unique.

Make a picture

Remedial assumption #2: Gale-Nikaido property (Gross substitutes, DD, etc.) or a unique consumer.

Attention: No pure intermediate goods:

One firm supply and the other demand iron (good i): For $y_{1i} = -y_{2i}$ any price clears the market. And for any $y_{1i} > -y_{2i}$ $p_i(\mathbf{y}) = 0$ and for $y_{1i} < -y_{2i}$ there is no equilibrium so $p_i(\mathbf{y})$ does not exist.

Now we have payoff functions

$$\pi_f(\mathbf{y}) = \mathbf{p}(\mathbf{y})\mathbf{y}_f, \mathbf{y}_f \in Y_f.$$

So we have a game in normal form $(N, Y_1, \dots, Y_n, \pi_1(\cdot), \dots, \pi_n(\cdot))$

A bonus of this assumption is that it implies that $\mathbf{p}(\cdot)$ is continuous ($\mathbf{p}(\cdot)$ is always upper-hemi continuous).¹

Problem #3: Payoffs depend on the normalization rule:

Explain why.

Why is this damaging? Because different shareholders may have different tastes $\rightarrow$ different interests $\rightarrow$ What should be maximized?

Remedial assumption: None (or to assume a large economy which takes away much of the punch). We blind ourselves and hope that nothing happens.

Problem #4: Convexity of Y_f

Remedial assumption #4: All production sets are convex

But imperfect competition is motivated by non-convexities!!

Silvestre "A model of general equilibrium with monopolistic behavior" *JET* 16, 1977

No monopsony, one output, MR and MC only intersect twice. Production function quasi-concave.

Then the best reply correspondence is a function, so it is continuous.

Input demand is uhc and convex valued so a fixed point exists.

At this point we should be brave and not being discouraged by just one more assumption. Harder times are ahead, just look at

Problem #5: Why Payoffs are quasi-concave on $\mathbf{y}_f$?

The "Anything goes" theorem suggest that this is not necessarily the case

¹Lack of continuity can be remedied in some cases by considering mixed strategy equilibria, see Dasgupta-Maskin "The Existence of Equilibrium in Discontinuous Economic Games, Parts I and II", Review of Economic Studies, 1-41, 1986. Dierker and Grodal (1986) showed an example in which mixed strategies do not help: "Nonexistence of Cournot-Walras equilibrium in a general equilibrium model with two oligopolists", in W. Hildenbrand and A. Mas Colell eds., Contributions to Mathematical Economics in Honor of Gérard Debreu, Amsterdam: North-Holland, 167-185.

Roberts and Sonnenschein "On the Foundations of the Theory of Monopolistic Competition," *Econometrica* 45, 1977.

Remedial assumption #5: Quasi concavity is imposed

Negishi (1961) "Monopolistic competition and general equilibrium" *REStud* 28, 1961. Conjectural Approach. The conjectured inverse demand function passes thru the real function and it is linear.

Other Fixed Point theorems that could be used:

Tarski: If $F : A \rightarrow A$ is increasing and A is a complete lattice it has a fixed point.² Make a picture. Similar problem; Why F is increasing?

In partial equilibrium with substitute goods, the best reply is increasing

McManus (1962) "Numbers and Size in Cournot Oligopoly" *Yorkshire Bulletin* 1962 rediscovered by Roberts and Sonnenschein *JET* 1976. Cumulative mapping $x_i = M(x)$ which comes from profit maximization. If marginal costs are increasing, $M(\cdot)$ is increasing, so by Tarski a symmetric equilibrium exists

Theorem 101 *Under the assumptions on consumers made in previous lectures, remedial assumptions 1-2-4-5 and if production sets are compact with $0 \in Y_f$, for all $f = 1, 2, \dots, n$ a Cournot-Nash exists.*

Proof. Define the best reply correspondence. It is u-h-continuous and convex valued so by Kakutani's fixed point theorem an equilibrium exists. ■

What about price-setting firms:

Some problems are alleviated.

We don't need to construct inverse demand functions:

Demand functions exist and are continuous

$x_i = D_i(\mathbf{p}, \mathbf{y})$ the Henri Ford effect. Assume it does not exist.

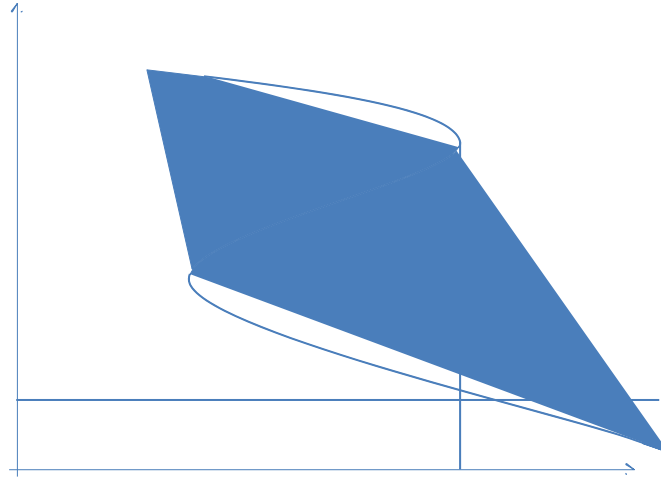
Payoffs depend on the normalization rule:

Convexity of Y_f

Payoffs are quasi-concave on $\mathbf{p}_f$?

Profits can be taken to be non negative.

²A complete lattice is a partially ordered set in which all subsets have both a supremum and an infimum.



10.3 Imperfect competition in general equilibrium: Limit economies

What about limit when the number of firms tend to infinite?

Kevin Roberts "The Limit points of monopolistic competition". *JET* 22, 1980.

Problem 1:

Two goods, good 1 is produced, good 2 is labor with marginal cost c .

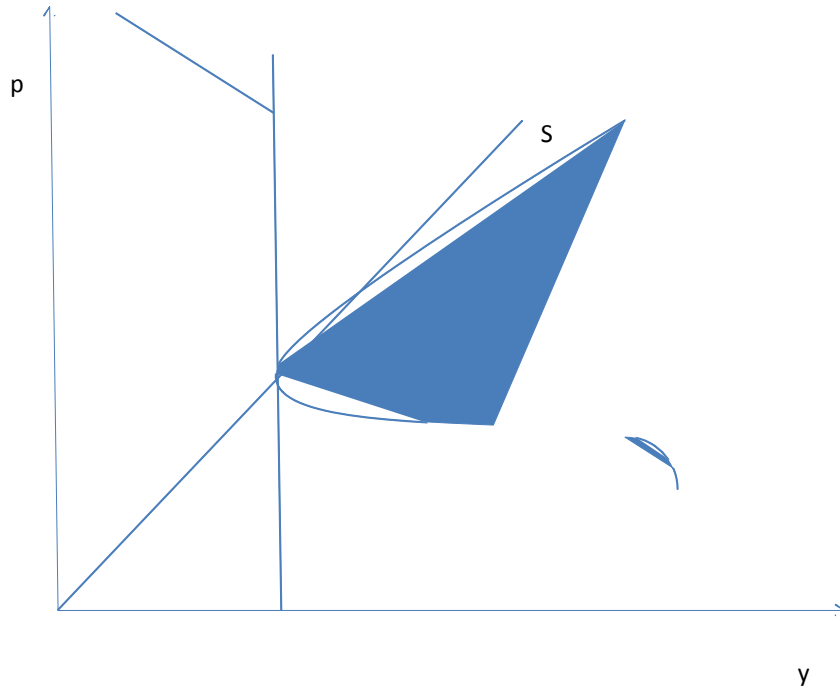
Walrasian correspondence as in Fig1 above.

The limit point when $n \rightarrow \infty$ is the critical economy far from the WE

Problem 2:

$z = (p - 2)^2 + 1 - y$ and supply function $S = p/2$.

Two WE $(p, y) = (2, 1), (2.5, 1.25)$. Both



We retain compact production sets with $0 \in Y_f$, for all $f = 1, 2, \dots, n$ and

1. There is a unique WE for any $\mathbf{y} \rightarrow \mathbf{p}(\mathbf{y})$ is continuous
2. Payoffs are strictly quasi-concave on $\mathbf{y}_f \rightarrow$ Best reply is a function
3. The global WE is unique.
4. Production sets and preferences are strictly convex $\rightarrow$ Supply and demand are functions
5. The economy consists of k firms and kn consumers which are identical.

Let E_k be the economy with k firms and kn consumers

Theorem 102 *Let $\mathbf{x}^k, \mathbf{y}^k, \mathbf{p}^k$ be a sequence of Cournot Nash equilibria. This sequence converges to the WE of the limit economy.*

Proof. First step: Since the allocation space is compact, there are always converging sequences by the Bolzano–Weierstrass theorem.

Second step: Since the choice of each type of agent are unique we can represent the choice of identical agents of type, say t by $\mathbf{x}^{kt}, \mathbf{y}^{kt}$.

Third step: The production plans of each type of firm converge to the production plans to the WE of the limit economy. Notice that prices depend on average output of each type of firms (like in (10)). And when the number of firm tends to infinite, the average is independent of output of any particular firm.

■

11 Other foundations of competitive equilibrium: Mechanism design. The core.

11.1 Mechanism design

What about designing a mechanism whose outcomes are Walrasian?

This is like creating a new material: Hurwicz (1917-2008).

Let A be the allocation set: fixed

$u_i : A \rightarrow \mathbb{R}$. $u_i \in U_i$. Let $U = \times_{j=1}^h U_j$.

Let e to be an economy $e = (u_1, u_2, \dots, u_h)$

Definition 103 *A social choice correspondence is a mapping $\phi : U \rightrightarrows A$.*

This mapping embodies social objectives.

People communicate in a certain language.

Let S_j be the set of messages of j and $S = \times_{j=1}^h S_j$

Let $g : S \rightarrow A$ the outcome function

Definition 104 *A mechanism M is a list $(S_1, S_2, \dots, S_h, g)$ where $g : S \rightarrow A$*

A mechanism is a language and the consequences of the messages.

A mechanism plus an economy induce a game in normal form:

$$(u_1(g(s)), u_2(g(s)), \dots, u_h(g(s))) \equiv (v_1(s), v_2(s), \dots, v_h(s))$$

A solution concept, given a mechanism M is a mapping $E_M : U \rightarrow S$

For instance Nash, Dominant Strategies, etc.

Definition 105 *A mechanism M implements the SCC ϕ if*

$$\forall u \in U, E_M(u) \neq \emptyset, g(E_M(u)) = \phi(u)$$

It turns out that if individual feasibility is ignored, there are very simple mechanisms implementing the Walrasian SCC in Nash and even in Nash and Strong equilibria.

11.1.1 Private Goods

Two approaches:

1. Abstract mechanisms

Example 106 *Hurwicz, REStud 1979.*

Example 107 *Schmeidler Econometrica 1980*

Messages for each agent $\mathbf{s}_j = (\mathbf{p}_j, \mathbf{z}_j)$ where $\mathbf{p}_j$ ($\mathbf{z}_j$) is the price (net trade) proposed by j .

Only messages such that $\mathbf{p}_j \mathbf{z}_j \leq 0$ and $\mathbf{p}_j \in S$ are allowed.

The outcome function is: $g_j = z_j - \sum \frac{z_h}{\#T_h}$ where T_h is the set of agents who sent identical prices.

Theorem 108 *The previous mechanism implements the Walrasian correspondence in Nash equilibrium when $n > 2$.*

Proof. Give agents the "Walrasian strategies" $(\mathbf{p}_j^W, \mathbf{z}_j^W)$ if somebody deviates in prices she cannot trade. So all agents are price-takers and the result follows. Conversely, let $(\mathbf{p}_j^*, \mathbf{z}_j^*)$ the strategies corresponding to a Nash equilibrium. ■

There are always two ideas:

- a) If you don't accept the prices set by others you have a penalty
- b) You can trade in all markets

- 2 Bertrand-like

Example 109 *Benassy (Econometrica 1986)*

11.1.2 Public goods

It is even more surprising that Lindahl equilibrium can be implemented

Suppose 1 public, 1 private good. $h \geq 3$

CRS in the production of the public good $y = \sum_{j=1}^n (w_j - x_j)$.

Walker proposed the following outcome functions:

$$y = \sum_{j=1}^h s_j \text{ and } x_j = w_j - (1/h + s_{j+2} - s_{j+1}) \sum_{j=1}^h s_j$$

Notice that this outcome function is balanced $\sum_{j=1}^h x_j = \sum_{j=1}^h w_j - y$

Has minimal dimensionality.

It is continuous.

But it does not necessarily picks up individually feasible allocations.

Proposition 110 *Walker mechanism implements the Walrasian correspondence in Nash equilibrium*

Proof. If s^* is a Nash equilibrium each individual maximizes $u_j(y, x_j)$ over $x_j = w_j - (1/h + s_{j-1}^* - s_{j+1}^*) \sum_{j=1}^h s_j$ so setting the Lindahl price for j as $q_j = 1/h + s_{j-1} - s_{j+1}$ we are done.

The converse follows from considering a Lindahl equilibrium $(y^L, \mathbf{x}^L, \mathbf{q}^L)$ with $\sum_{j=1}^h q_j^* = 1$ and solving the following system of equations

$$\begin{aligned} \sum_{j=1}^h s_j &= y^L \\ s_2 - s_1 &= q_h^L - 1/h \\ &\dots \\ s_h - s_{h-1} &= q_1^L - 1/h \end{aligned}$$

We can show that the determinant of this system has value n so the system has a solution and this solution is a Nash equilibrium (notice that $s_1 = s_h + q_{h-1} - 1/h$.

■

Bevia-Corchon-Wilkie and Corchon-Wilkie

11.2 The core

No institutional agreement other than rule of law

People are complete free to trade.

A coalition C is a non empty set of players

For simplicity we assume an exchange economy

Definition 111 An allocation $(\mathbf{x})$ is a list of consumptions $\mathbf{x}_j \in X_j$ $j = 1, 2, \dots, h$

Definition 112 An allocation $(\mathbf{x})$ is feasible if for all $i = 1, 2, \dots, m$,

$$\sum_{j=1}^h (x_{ji} - w_{ji}) \leq 0.$$

Definition 113 The core is a feasible allocation (x^C) such that there is no coalition C and no x'_j , $j \in C$ such that

a) $\sum_{j \in C} (x'_{ji} - w_{ji}) \leq 0.$

b) $u_j(x'_j) > u_j(x_j^P)$ for all $j \in C$

Remark 114 An alternative definition of the Core are obtained by substituting the inequalities $>$ by $\geq$ with, at least, one inequality strict. If preferences are strictly monotonic, both definitions are equivalent.

Remark 115 Any core allocation is Pareto efficient.

Definition 116 A Walrasian (or perfectly competitive) equilibrium is a feasible allocation (x^W) and a vector of prices p^W such that:

For all $j = 1, 2, \dots, h$ x_j^W maximizes utility on the set $B_j(p^W, y^W)$.

Proposition 117 *The Walrasian equilibrium allocation is in the core*

Proof. Suppose not.... identical to the first welfare theorem ■

The converse is much more involved

And needs new assumptions in addition to the classical ones

Consumption set is $\mathbb{R}_+^n$

Preferences are strictly convex and strictly monotone

Start with an economy with h types.

Define a replica

Lemma 118 *In any core allocation, all consumers with the same type get the same consumption bundle.*

Proof. Suppose not. In particular let j and k be two identical consumers that get different bundles. Consider a coalition of the worse off individuals of each type. Consider the consumption that is the average consumption for each type. This is feasible and preferred at least by either j or k and indifferent or preferred by everybody else in the coalition. ■

From then on we can represent the consumption of each type, say j , by a vector $\mathbf{x}_j$

So an allocation is like in the replica 1 economy

To simplify we make some additional assumptions:

Differentiability of preferences

All core allocations are interior

Lemma 119 *If an allocation $\mathbf{x}$ is in the core for all replicas, this allocation is a WE*

Proof. Take $\mathbf{x} \gg \mathbf{0}$ and Pareto efficient. So by the second welfare theorem there are prices such that $\mathbf{x}$ is a competitive equilibrium with prices $\mathbf{p}$. If the allocation is not Walrasian it must be that $\mathbf{p}\mathbf{x}_j > \mathbf{p}\mathbf{w}_j$ some j . Now consider a coalition of all players except j . This coalition has to absorb $\mathbf{x}_j - \mathbf{w}_j$.

Notice that

$$\mathbf{p}(\mathbf{x}_j - \mathbf{w}_j) = \frac{\partial u_k}{\partial \mathbf{x}_k}(\mathbf{x}_j - \mathbf{w}_j)$$

Notice that

$$du_k \simeq \sum_{i=1}^n \frac{\partial u_k}{\partial x_{ki}} dx_{ki}$$

so if the number of agents is large enough $\mathbf{x}_j - \mathbf{w}_j \simeq d\mathbf{x}_j$ and the theorem is proved. ■

Remark 120 *Replication of all types is essential. Example of Gabsewicz.*

Proof. Suppose $x_j \succ_j x_k$. Clearly $x_j \neq w_j$ because if not k will improve upon by herself. Then k can join a coalition with everybody else and offer them to take only $x_j - \epsilon$ which everybody prefers. If $x_j \sim_j x_k$ ■